A Markovian Approach to Darts

Model and Simulation

Bortolon, F. Castiglione, C.
Parolini, L. Schiavon, L.

Università degli Studi di Padova

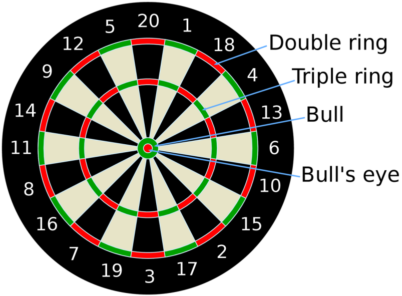
27 June 2017

Table of contents

Game Introduction

Rules and dartboard

- Initial total score is 501 points;
- at each round, the player throws 3 darts and subtracts the scores;
- the match ends when someone gets to 0, necessarily with a double score at the last throw;
- no score is given for throws that lead to negative score or to 1.



Distribution of the single throw

Dart motion

$$\begin{cases} d = v_x t \\ y = v_y t \\ z = v_z t - \frac{g}{2} t^2. \end{cases}$$

We assume that the velocity component v_x is fixed; in our single throw simulation we choose the value $v_x = 17882$ mm/s. The player introduces an error; we suppose that:

$$\begin{cases} v_y = \overline{v_y} + \epsilon_y, & \epsilon_y \sim \mathcal{N}(0, \sigma_y^2) \\ v_z = \overline{v_z} + \epsilon_z, & \epsilon_z \sim \mathcal{N}(0, \sigma_z^2) \end{cases}$$

Distribution on the Dartboard

The coordinates y, z hit at each shooting have normal indipendent distribution:

$$y \sim \mathcal{N}\left(\overline{v_y}t, t^2\sigma_y^2\right)$$
$$z \sim \mathcal{N}\left(\overline{v_z}t - \frac{g}{2}t^2, t^2\sigma_z^2\right).$$

This allows to write the joint density function of the polar coordinates ρ, θ as a bivariate normal:

$$f_{\overline{\rho},\overline{\theta}}(\rho,\theta) = \rho\frac{1}{2\pi\sigma_y\sigma_z t^2} exp\left\{-\frac{(\rho\, cos\theta - \overline{v_y}t)^2}{2t^2\sigma_y^2} - \frac{(\rho\, sin\theta - \overline{v_z}t + \frac{g}{2}t^2)^2}{2t^2\sigma_z^2}\right\}.$$

Strategy

Score optimization on a single throw

Score optimization

A convenient strategy during the first part of the match is to decrease the score as fast as possible. The expected value of the score s, conditioned to the target velocity $\overline{v_y}$ and $\overline{v_z}$, is:

$$E_{\overline{v_y},\overline{v_z}}(s) = \int_{R_1 \cup \ldots \cup R_{83}} S(\rho,\theta) f_{\overline{\rho},\overline{\theta}}(\rho,\theta)\, d\rho\, d\theta =$$

$$= \sum_{i=1}^{83} S_i \int_{R_i} f_{\overline{\rho},\overline{\theta}}(\rho,\theta)\, d\rho\, d\theta$$

Score optimization

We associate at each possible target region on the dartboard a target velocity $(\overline{v_y}, \overline{v_z})$ and we maximize the expected value, varying the latter.
The strategy chosen corresponds to $(\overline{v_y}, \overline{v_z})^j$ such that:

$$E_{(\overline{v_y},\overline{v_z})^j}(s) = \max_{i \in I} E_{(\overline{v_y},\overline{v_z})^i}(s).$$

Strategy

Closing process

Markov process

At 170 points begins the final part of the match, when it is possible to win using at most 3 darts, except a few cases (e.g., 159, 169, ...). We model this part with a stochastic process:

- Let $E = \{0, 2, 3, 4, \ldots 170\}$ be the state space.
- Let $S = \{1, 2, 3, \ldots, 20\} \cup \{2, 4, 6, \ldots, 40\} \cup \{3, 6, 9, \ldots, 60\} \cup \{25, 50\}$ be the set of all scores achievable with a single throw.
- Let $\{X_n\}_{n \geqslant 0}$ be the stochastic process that describes the total score achieved after the n–th throw. We define X_0 as the maximum total score s such that $s \leqslant 170$.

The process $\{X_n\}_{n \geqslant 0}$ is a
discrete time Markov chain with state space E.

Transition matrix

- $p_{0j} = \delta_{0j}$ since 0 is an absorbing state;
- $p_{ij} > 0$ if and only if it exists $s \in S$ such that $j = i - s$ and i even if $j = 0$;
- $p_{ij} = 0$ when $i < j$ as follows from the previous point. Notice that P is a lower triangular matrix;
- given $i \in E$ and a target score $s \in S$, we find p_{ij} integrating the density function relative to s on the regions that give $i - j$ score.

This works if we have a unique strategy for the each state i, but how do we choose strategies?

Optimal stategy

We want to define an optimal strategy for each state i:

- if it is not possible to win with at most 3 throws we maximize the average score as seen previously; e.g. $i = 169, 159, \ldots;$
- if $i \in \{2, 4, 6, \ldots, 40, 50\}$ is a state that admits a single-throw closing strategy we choose that one as optimal.

Optimal stategy

- Otherwise we choose as optimal strategy h a 2 or 3 throws one which generates an i–th row in the transition matrix that gives the minimum average absorbing time $k_i^{\{0\}}$ in $\{0\}$ from the state i.

This choice leads to the following linear systems of equations, varying the strategy h:

$$\begin{cases} k_l^{\{0\},\,h} = 1 + \sum_{j \in E} p_{lj}^h k_j^{\{0\},\,h} \\ k_0^{\{0\},\,h} = 0 \end{cases} \qquad l = 0, 1, \ldots, i.$$

We choose the strategy h^* which gives the minimum solution:

$$k_i^{\{0\},\,h^*} = \min_h \left\{ k_i^{\{0\},\,h} \right\}.$$

Multi-player hypotesis

Sometimes a player would choose, if possible, a strategy that allows to win in a 3-throws round, even if it is not the best one relative to the average absorbing time.

We define new states as couples $(i; j)$, where:

- i is the total score;
- $j = 1, 2, 3$ is the number of remaining darts.

with i the total score and $j = 1, 2, 3$ the remaining darts in the current round.

If $\alpha := (i, j)$ $\beta := (l, h)$ are two states the transition probability $p_{\alpha\beta}$ is non zero if and only if we have same conditions as before on the first component about the score and also $h = j - 1$ or $h = 3$ if $j = 1$.

Multi-player hypotesis

Each equation of the previous syestems is replaced by 3 new ones:

$$\begin{cases} k_{l,1}^{\{0\},\,h} = 1 + \sum\limits_{j\in E} p^{h}_{(l,1)(j,3)} k_{j,3}^{\{0\},\,h} \\ k_{l,2}^{\{0\},\,h} = 1 + \sum\limits_{j\in E} p^{h}_{(l,2)(j,1)} k_{j,1}^{\{0\},\,h} \\ k_{l,3}^{\{0\},\,h} = 1 + \sum\limits_{j\in E} p^{h}_{(l,3)(j,2)} k_{j,2}^{\{0\},\,h}, \end{cases}$$

In this way it is possible to compare all different combinations of three strategies. So we can impose that a player in a state $(i, 2)$ chooses only one-throw or two-throws strategies, if any exists.

Simulation Results

Simulation

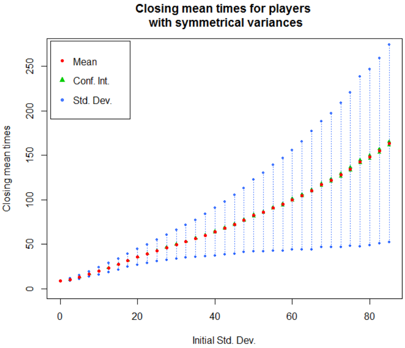
- Each player is characterised by variances on v_y and v_z, that represent precision.
- For each player we computed the transition matrix.
- For each case we simulated 10000 matches with *Monte Carlo* method.

Some results of simulations

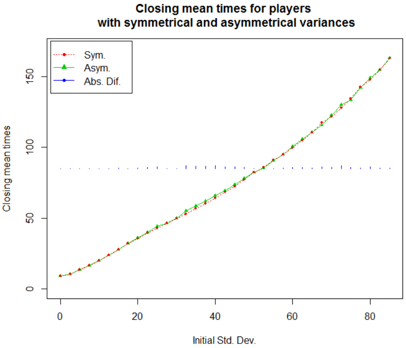
R. mm	Average Throws	Std. Dev. Throws^2	K max Throws	K mean Throw	K min Throws	P bullseye Throws	P eye	BD p	BD t
5	10,25	1,06	4,11	2,47	1,08	0,9603	1,0000	60	3
10	13,28	1,99	5,32	3,48	1,74	0,5536	0,9940	60	3
15	16,46	2,91	6,83	4,55	2,46	0,3012	0,8973	60	3
20	19,81	3,93	8,44	5,68	3,25	0,1826	0,7220	60	3
25	23,61	5,03	10,21	6,97	4,14	0,1211	0,5592	60	3
40	35,68	8,74	16,56	11,95	7,81	0,0492	0,2739	57	3
55	46,26	14,15	24,03	18,53	13,16	0,0263	0,1557	57	3
70	56,58	20,74	32,71	26,67	20,21	0,0163	0,0992	21	3
95	77,09	35,90	50,73	43,93	35,73	0,0089	0,0552	21	3
120	99,86	55,73	72,97	65,75	55,97	0,0056	0,0349	50	2
145	127,86	80,83	100,29	92,56	80,91	0,0038	0,0241	50	2
170	162,99	110,82	133,01	124,57	110,56	0,0028	0,0176	50	2

Comparison between optimal strategies

Score	Sym Variance Player			Asym Variance Player		
	1-throw st.	2-throw st.	3-throw st.	1-throw st.	2-throw st.	3-throw st.
54	T 14	T 14	T 14	T 11	T 14	T 11
55	T 17	T 17	T 17	19	19	19
56	T 16	T 16	T 16	20	20	20
57	T 17	T 17	T 17	T 17	T 17	T 17
58	T 18	T 18	T 18	T 18	T 18	T 18
59	T 19	T 19	T 19	T 19	T 19	T 19
60	T 16	T 16	20	T 8	T 8	T 8
61	T 19	T 19	T 19	T 19	T 19	T 19
62	T 14	T 14	T 14	T 14	T 14	T 14
63	T 14	T 11	D 25	T 14	T 11	T 14
64	T 16	T 16	T 16	T 8	T 8	T 8
65	T 19	T 19	T 19	T 14	T 11	T 14
66	T 16	T 16	T 16	T 16	T 16	T 16
67	T 14	T 11	T 14	T 11	T 11	T 11
68	T 14	T 14	T 9	T 14	T 14	T 14
69	T 7	T 19	T 7	T 11	T 11	T 11
70	T 14	T 14	T 14	T 11	T 14	T 11
71	T 7	T 7	T 7	T 7	T 7	T 7
72	T 16	T 16	T 16	T 8	T 16	T 8
73	T 19	T 19	T 19	T 14	T 11	T 14
74	T 7	T 16	T 7	T 7	T 14	T 7
75	T 7	T 19	T 7	T 8	T 19	T 8

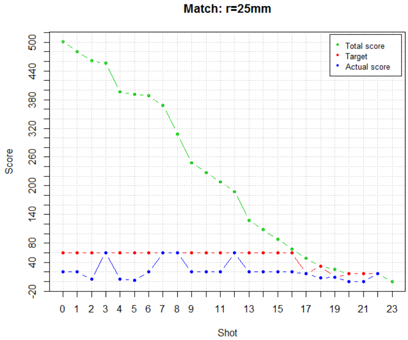

Closing mean times for players
with symmetrical variances
Mean
Conf. Int.
Std. Dev.
Closing mean times
Initial Std. Dev.
0
50
100
150
200
250
0
20
40
60
80

(a)

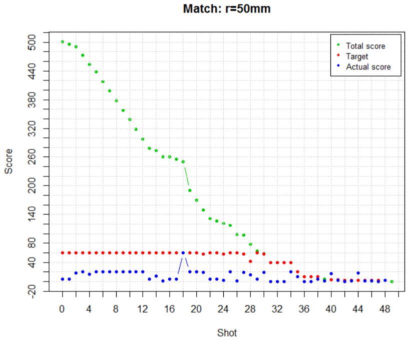

Closing mean times for players
with symmetrical and asymmetrical variances
Sym.
Asym.
Abs. Dif.
Closing mean times
Initial Std. Dev.
0
50
100
150
0
20
40
60
80

(b)

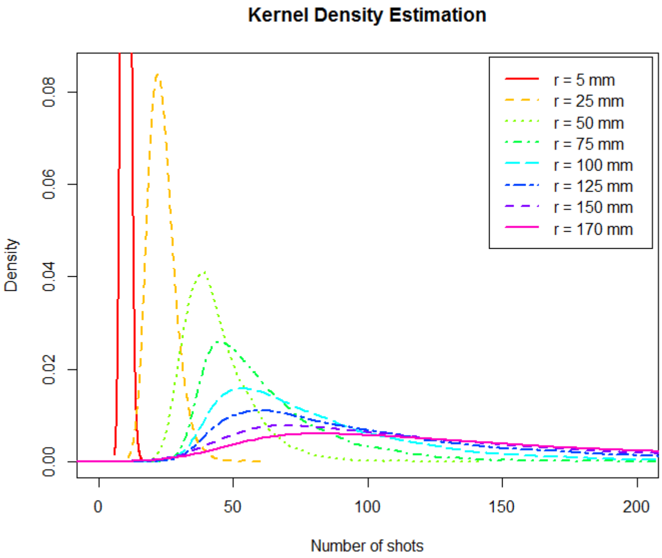
Example of single match



(c)



(d)


Kernel Density Estimation
r = 5 mm
r = 25 mm
r = 50 mm
r = 75 mm
r = 100 mm
r = 125 mm
r = 150 mm
r = 170 mm
Density
0.00
0.02
0.04
0.06
0.08
0
50
100
150
200
Number of shots

A simulation with empirical data

The Markovian approach to darts matches increases the performances; to show that we give an example of how our algorithm works if the transition matrix is empirically obtained.

These data were collected by repetition of 100 throws for every target on the dartboard.

We compared with a simulation the performances of 2 different players:

- *Smart* player: uses a Markovian approach;
- *Naive* player, uses standard strategies.

A simulation with empirical data

	M throws	95% CI for M throws	sd throws	P	95% CI for P
Smart Player	32.19	(32.03; 32.35)	7.98	0.623	(0.614; 0.632)
Naive Player	36.38	(36.20; 36.56)	9.01		

References

- http://www.dartbase.com/Sect1/17.html.
- R. J. Tibshirani, A. Price, J. Taylor. A Statistician Plays Darts. *Journal of the Royal Statistic Society*, 174: pp.213-226, 2011.
- James R. Norris *Markov Chains.* Cambridge University Press, 1997.
- Sheldon M. Ross *A First Course in Probability.* Pearson Prentice Hall, 1976.
- J. Jacod, P. Protter *Probability Essentials.* Springer, 1999.
- C. P. Robert, G. Casella *Introducing Monte Carlo Methods with R.* Springer, 2010.
- D. Kleppner, R. Kolenkow *An Introduction to Mechanics.* Cambridge University Press, 1973.