On Reducing Sequence Effects in Competitions

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Many sports include a phase where teams or athletes take turns performing a task and the winner is the one who has the most successes. Examples include penalty shots where a soccer game results in a tie and a winner has to be declared, playoff series in various sports where a "turn" is a home game, as well as the manner of playing an overtime period in a tied American football game. The sequence in which the attempts are played in many of these tie-breakers has a real or perceived effect on the outcome.

A common way to determine the order or sequence in such situations is by coin-toss. However, while that constitutes ex-ante fairness, it does not generate ex-post fairness, and it is considered rather undesirable to allow a coin-toss to significantly affect the outcome of a match (Brams and Sanderson 2013).

In tied soccer games that require a clear winner, teams typically take turns shooting a total of 5 penalty shots each from a distance of 11m: *ABABABABAB*. Similar tie breakers are now used in the NHL in every tied game. Many fans and sports reporters believe team *A* to have a "psychological advantage" in this shoot-out and there is some empirical research supporting that. Other empirical research, however, refutes those findings.

We consider an idea of *bidding* for the place in the sequence by the difficulty of the task. In soccer, assuming for simplicity that each team is allowed only one attempt (kick), the bid could be for the distance of its kicking point from the goal. If *A* bids 12.5m and *B* 11.5m then *A* would kick first, from a distance of some average of 12.5m and 11.5m, while *B* would kick second from either 11m ("Rule 1") or from an average of 11.5m and 12.5m ("Rule 2").

That is an auction with positive externality, since the team with losing bid is still influenced (positively) by the magnitude of the winning bid.

We analyze the resulting games for both rules, for discrete and continuous ability distributions. We then compare the bids resulting from each rule.

Suppose the teams are evenly matched, risk neutral and characterized by their abilities to score on a penalty kick, denoted by x for A and y for B, $x, y \in [0, \infty)$. Each team knows its own ability and forms a probability distribution function, $F(\cdot)$, over the other team's ability. Assume that if both attempts were from 11m, the following success probabilities for a team with ability y are known to be:

$p(y)$ – the probability that the first attempt is successful;

$q(y)$ – the probability that the second attempt is successful if the first attempt by the other team was successful;

$r(y)$ – the probability that the second attempt is successful if the first attempt by the other team failed.

One would expect that

$$p(y) > r(y) > q(y) \ \forall y.$$

The probabilities decrease with the kick distance. If a team with ability y attempts from a distance (11+z)m, then we assume that the probability of success is $p(y)e^{-\theta z}$, where $\theta > 0$ is given. The other probabilities, $q(y)$ and $r(y)$, change similarly.

Suppose the teams follow a symmetric strategy that determines the bid a team submits given its ability, $\beta(y)$, $\beta:[0,\infty)\rightarrow[0,\infty)$. We seek a monotone increasing strategy $\beta(y)$, which maximizes the expected probability that A wins. Note that we focus on an outright win – a tie is given no value.

We denote the bids of teams A and B by a and b, respectively, and assume that if $a > b$, then team A attempts first from a distance of $\alpha a + (1 - \alpha)b$, $\frac{1}{2} \leq \alpha \leq 1$, where α is fixed by the organizers and known to the teams. As far as the team with a lower bid is concerned, Rule 1 has it attempting a shot from 11m (i.e., zero extra distance), while Rule 2 requires it to attempt its shot from a distance of $\alpha b + (1 - \alpha)a$, with the same α as used for the team attempting first. If $a < b$ the roles of the teams are reversed.

Rule 1

The objective of team A is to maximize its expected probability of winning,

$$\max_{\beta(\cdot)} J$$
$$= \int_0^x p(x)e^{-\theta\left(\alpha\beta(x)+(1-\alpha)\beta(y)\right)}\left(1-q(y)\right)f(y)dy$$
$$+ \int_x^\infty \left(1-p(y)e^{-\theta(\alpha\beta(y)+(1-\alpha)\beta(x))}\right)r(x)f(y)dy$$

Two values distribution of abilities:

Suppose that

$$X, Y = \begin{cases} c & \text{with probability } \varepsilon \\ d & \text{with probability } 1-\varepsilon \end{cases} \qquad 0 \le \varepsilon \le 1,$$

and without loss of generality $c > d$.

Suppose that in case of ties the winner is selected randomly.

Suppose first that $x = c$, i.e., $\beta(c) = a$, then,

$$J = \frac{\varepsilon}{2} p(c) e^{-\theta a}\big(1 - q(c)\big) + \frac{\varepsilon}{2} r(c)\big(1 - p(c)e^{-\theta a}\big)$$
$$+ (1-\varepsilon) p(c) e^{-\theta(\alpha a + (1-\alpha)\beta(d))}\big(1 - q(d)\big)$$

Since J decreases in $\beta(d)$ for each a, then the value of $\beta(d)$ that maximizes J is $\beta(d) = 0$. After substituting $\beta(d) = 0$ the maximization of J

$$J = \frac{\varepsilon}{2} p(c) e^{-\theta a}\big(1 - q(c)\big) + \frac{\varepsilon}{2} r(c)\big(1 - p(c)e^{-\theta a}\big)$$
$$+ (1-\varepsilon) p(c) e^{-\theta \alpha a}\big(1 - q(d)\big)$$

w.r.t. parameter a, is carried out by solving the equation $\frac{d}{da} J = 0$. The result is,

$$a = \begin{cases} \frac{1}{\theta(1-\alpha)} ln \frac{\varepsilon(q(c) + r(c) - 1)}{2\alpha(1-\varepsilon)\big(1 - q(d)\big)}, & if \quad q(c) + r(c) > 1 + \frac{2\alpha(1-\varepsilon)\big(1-q(d)\big)}{\varepsilon} \\ 0, & otherwise \end{cases}$$

Suppose now that $x = d$, i.e., $\beta(d) = a$, then,

$$J = \frac{1-\varepsilon}{2} p(d) e^{-\theta a}\big(1 - q(d)\big) \frac{1-\varepsilon}{2} r(d)\big(1 - p(d) e^{-\theta a}\big)$$
$$+\varepsilon\big(1 - p(c) e^{-\theta(\alpha\beta(c)+(1-\alpha)a)}\big) r(d).$$

Since J increases in $\beta(c)$ for each a, then the value of $\beta(c)$ that maximizes J is $\beta(c) = \infty$. Substituting, the bid a that maximizes the objective J,

$$J = \frac{1-\varepsilon}{2} p(d) e^{-\theta a}(1 - q(d) - r(d)) + \frac{1+\varepsilon}{2} r(d),$$

is

$$a = \begin{cases} \infty, & if \quad q(d) + r(d) > 1 \\ 0, & otherwise \end{cases}$$

Continuous exponential distribution of abilities:

Suppose that the beliefs of the teams with regard to the abilities of the other team are distributed exponentially, $X, Y \sim Exp(\lambda)$ and the probabilities $p(y)$, $q(y)$ and $r(y)$ grow with the team ability in the form,

$$p(y) = 1 - e^{-py}\ ,\ \ q(y) = 1 - e^{-qy},\ \ r(y) = 1 - e^{-ry}, \quad p > r > q.$$

This section limits the solution of the problem to the linear $\beta(\cdot)$,

$$\beta(y) = ty$$

which is simple to implement. In such a case,

$$J = \lambda e^{-\lambda x}\left(\frac{e^{-(q+t\theta)x}(1 - e^{-px})(e^{(\lambda+q+t(1-\alpha)\theta)x} - 1)}{\lambda + q + t(1-\alpha)\theta} - \frac{e^{-t\theta x}(1 - e^{-rx})}{\lambda + t\alpha\theta} + \frac{1 - e^{-rx}}{\lambda} + \frac{e^{-(p+t\theta)x}(1 - e^{-rx})}{\lambda + p + t\alpha\theta}\right)$$

In the numerical experiment below, we use $\theta = 0.1$, $p = 0.1$, $r = 0.09$, $q = 0.08$ and $x = 17$, which imply that the probabilities of success from 11m are $p(x) \approx 0.82$, $r(x) \approx 0.78$, and $q(x) \approx 0.74$. Also, take $\alpha = 0.5$. The results of this case are presented in the following figures. We observe that team's bid increases with the expected ability of the other team. However, the dependence of a team's bid on its own ability differs whether Rule 1 or Rule 2 is applied: the bid decreases for Rule 1 and increases for Rule 2.

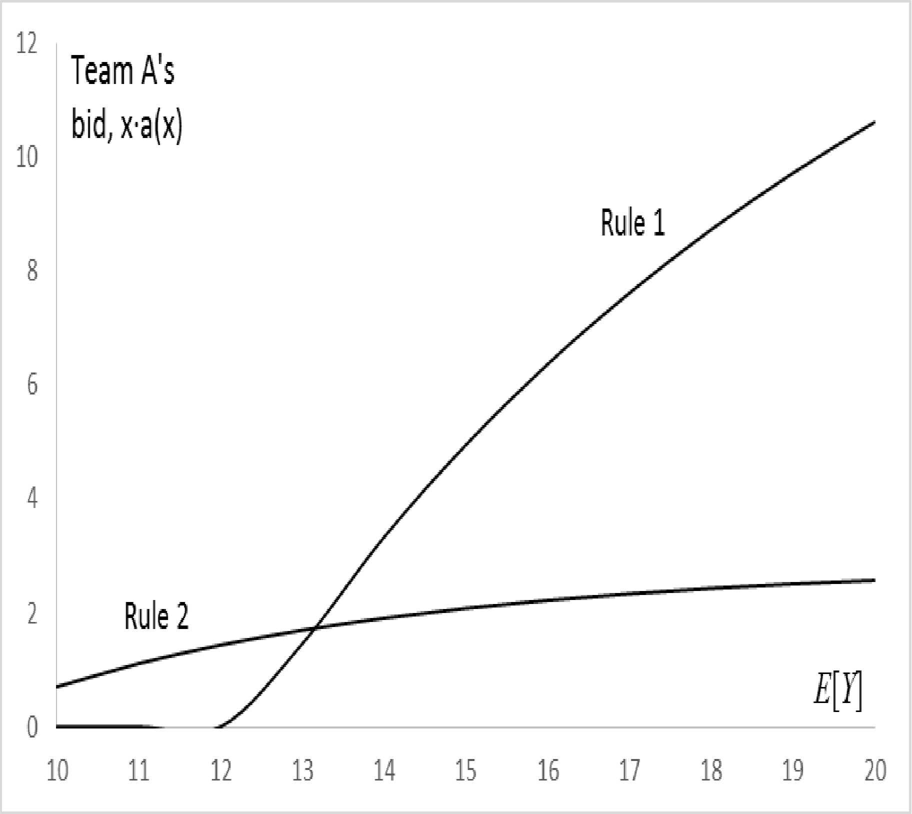


Figure 1

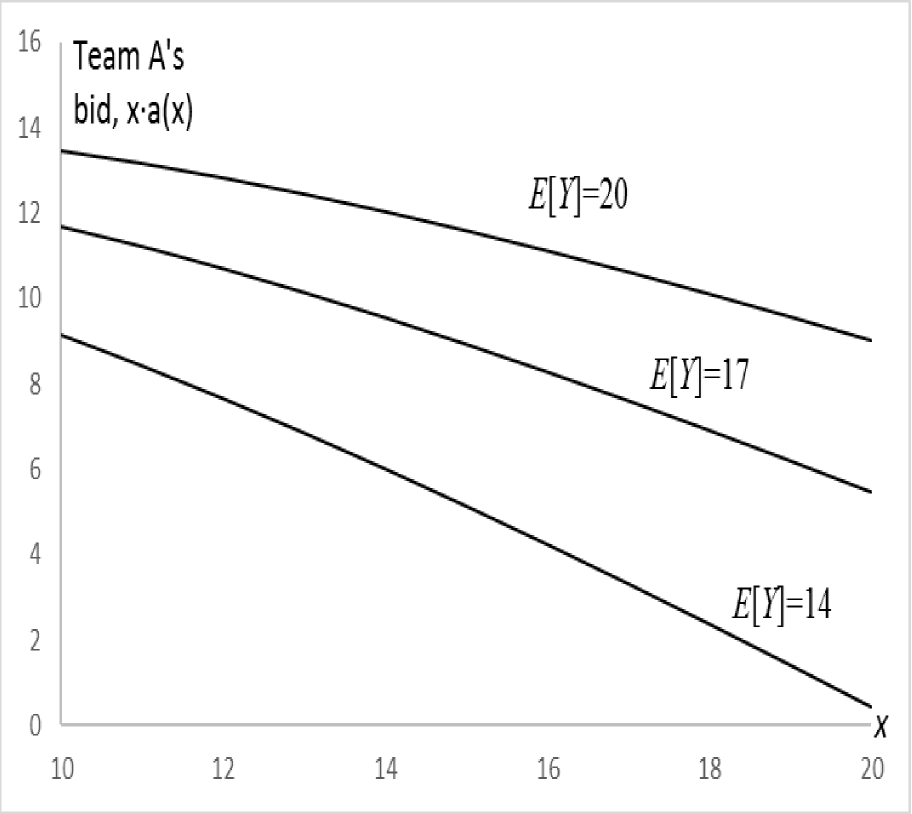


Figure 2

Rule 2

This rule requires both teams to shoot from a distance which is a linear combination of the two bids. The expected probability of *A* winning is now,

$$\max_{\beta(\cdot)} J$$
$$= \int_0^x p(x) e^{-\theta\left(\alpha\beta(x)+(1-\alpha)\beta(y)\right)}\left(1-q(y)e^{-\theta\left(\alpha\beta(y)+(1-\alpha)\beta(x)\right)}\right)f(y)dy$$
$$+\int_x^\infty \left(1-p(y)e^{-\theta(\alpha\beta(y)+(1-\alpha)\beta(x))}\right)r(x)e^{-\theta(\alpha\beta(x)+(1-\alpha)\beta(y))}f(y)dy\,.$$

Two values distribution of abilities:

Suppose that $x = c$, i.e., $\beta(c) = a$. Then,

$$J = \frac{\varepsilon}{2} p(c) e^{-\theta a}\left(1 - q(c)e^{-\theta a}\right) + \frac{\varepsilon}{2} r(c) e^{-\theta a}\left(1 - p(c)e^{-\theta a}\right)$$
$$+(1-\varepsilon)p(c)e^{-\theta(\alpha a+(1-\alpha)b)}\left(1 - q(d)e^{-\theta(\alpha b+(1-\alpha)a)}\right).$$

By solving $\frac{d}{db}J = 0$ w.r.t. b, we get that b depends linearly on a,

$$b^*(a) = \frac{1}{\alpha\theta} ln \frac{q(d)}{1-\alpha} - a\frac{1-\alpha}{\alpha}.$$

Substituting, we obtain

$$J = \frac{\varepsilon}{2} p(c) e^{-\theta a}\left(1 - q(c) e^{-\theta a}\right) + \frac{\varepsilon}{2} r(c) e^{-\theta a}\left(1 - p(c) e^{-\theta a}\right)$$
$$+(1-\varepsilon)\alpha p(c)\left(\frac{1-\alpha}{q(d)}\right)^{\frac{1-\alpha}{\alpha}} e^{-\theta a \frac{2\alpha-1}{\alpha}}.$$

After changing variable, $z \equiv e^{-\theta a}$,

$$J = c_1 z - c_2 z^2 + c_3 z^{\frac{2\alpha-1}{\alpha}},$$

where

$$c_1 = \frac{\varepsilon}{2}\left(p(c) + r(c)\right),\ \ c_2 = \frac{\varepsilon}{2} p(c)\left(q(c) + r(c)\right)$$

and $c_3 = (1-\varepsilon)\alpha p(c)\left(\frac{1-\alpha}{q(d)}\right)^{\frac{1-\alpha}{\alpha}}$.

In the particular case $\alpha = 1/2$, J is maximized at $z = \frac{c_1}{2c_2}$. That is,

$$a = \begin{cases} \frac{1}{\theta} ln \frac{2p(c)\left(q(c)+r(c)\right)}{p(c)+r(c)}, & if \;\; q(c)+r(c) > \frac{1}{2}\left(1+\frac{r(c)}{p(c)}\right) \\ 0, & otherwise \end{cases}.$$

In the particular case $\alpha = 1$, J is maximized at $z = \frac{c_1+(1-\varepsilon)p(c)}{2c_2}$. That is,

$$a = \begin{cases} \frac{1}{\theta} ln \frac{2\varepsilon p(c)\left(q(c)+r(c)\right)}{2p(c)-\varepsilon(p(c)-r(c))}, & if \;\; q(c)+r(c) > \frac{1}{\varepsilon} - \frac{1}{2}\left(1-\frac{r(c)}{p(c)}\right) \\ 0, & otherwise \end{cases}.$$

Suppose now that $x = d$, i.e., $\beta(d) = a$. Then,

$$J = \frac{1-\varepsilon}{2} p(d) e^{-\theta a}\left(1 - q(d) e^{-\theta a}\right) + \frac{1-\varepsilon}{2} r(d) e^{-\theta a}\left(1 - p(d) e^{-\theta a}\right)$$

$$+\varepsilon\left(1 - p(c) e^{-\theta(\alpha b + (1-\alpha) a)}\right) q(d) e^{-\theta(\alpha a + (1-\alpha) b)}.$$

By solving $\frac{d}{db} J = 0$, we get that b again depends linearly on a,

$$b^*(a) = \frac{1}{\alpha\theta} ln \frac{p(c)}{1-\alpha} - a\frac{1-\alpha}{\alpha}.$$

Substituting, we obtain

$$J = \frac{1-\varepsilon}{2} p(d) e^{-\theta a}\left(1 - q(d) e^{-\theta a}\right) + \frac{1-\varepsilon}{2} r(d) e^{-\theta a}\left(1 - p(d) e^{-\theta a}\right)$$

$$+\varepsilon\alpha q(d) \left(\frac{1-\alpha}{p(c)}\right)^{\frac{1-\alpha}{\alpha}} e^{-\theta a \frac{2\alpha-1}{\alpha}}.$$

After changing variable, $z \equiv e^{-\theta a}$,

$$J = c_1 z - c_2 z^2 + c_3 z^{\frac{2\alpha-1}{\alpha}},$$

where $c_1 = \frac{1-\varepsilon}{2}(p(d) + r(d))$, $c_2 = \frac{1-\varepsilon}{2} p(d)(q(d) + r(d))$ and

$c_3 = \varepsilon\alpha r(d) \left(\frac{1-\alpha}{p(c)}\right)^{\frac{1-\alpha}{\alpha}}$.

In the particular case $\alpha = 1/2$, J is maximized at $z = \frac{c_1}{2c_2}$. That is,

$$a = \begin{cases} \frac{1}{\theta} ln \frac{2p(d)\left(q(d) + r(d)\right)}{p(d) + r(d)}, & if \;\; q(d) + r(d) > \frac{1}{2}(1 + \frac{r(d)}{p(d)}) \\ 0, & otherwise \end{cases}$$

In the particular case $\alpha = 1$, J is maximized at $z = \frac{c_1 + \varepsilon r(d)}{2c_2}$. That is,

$$a = \begin{cases} \frac{1}{\theta} ln \frac{2(1-\varepsilon)p(d)\left(q(d)+r(d)\right)}{2\varepsilon r(d) + (1-\varepsilon)(p(d)+r(d))}, & if \;\; q(d) + r(d) > \frac{1}{2}(1 + \frac{r(d)}{p(d)}\frac{1+\varepsilon}{1-\varepsilon}) \\ 0, & otherwise \end{cases}.$$

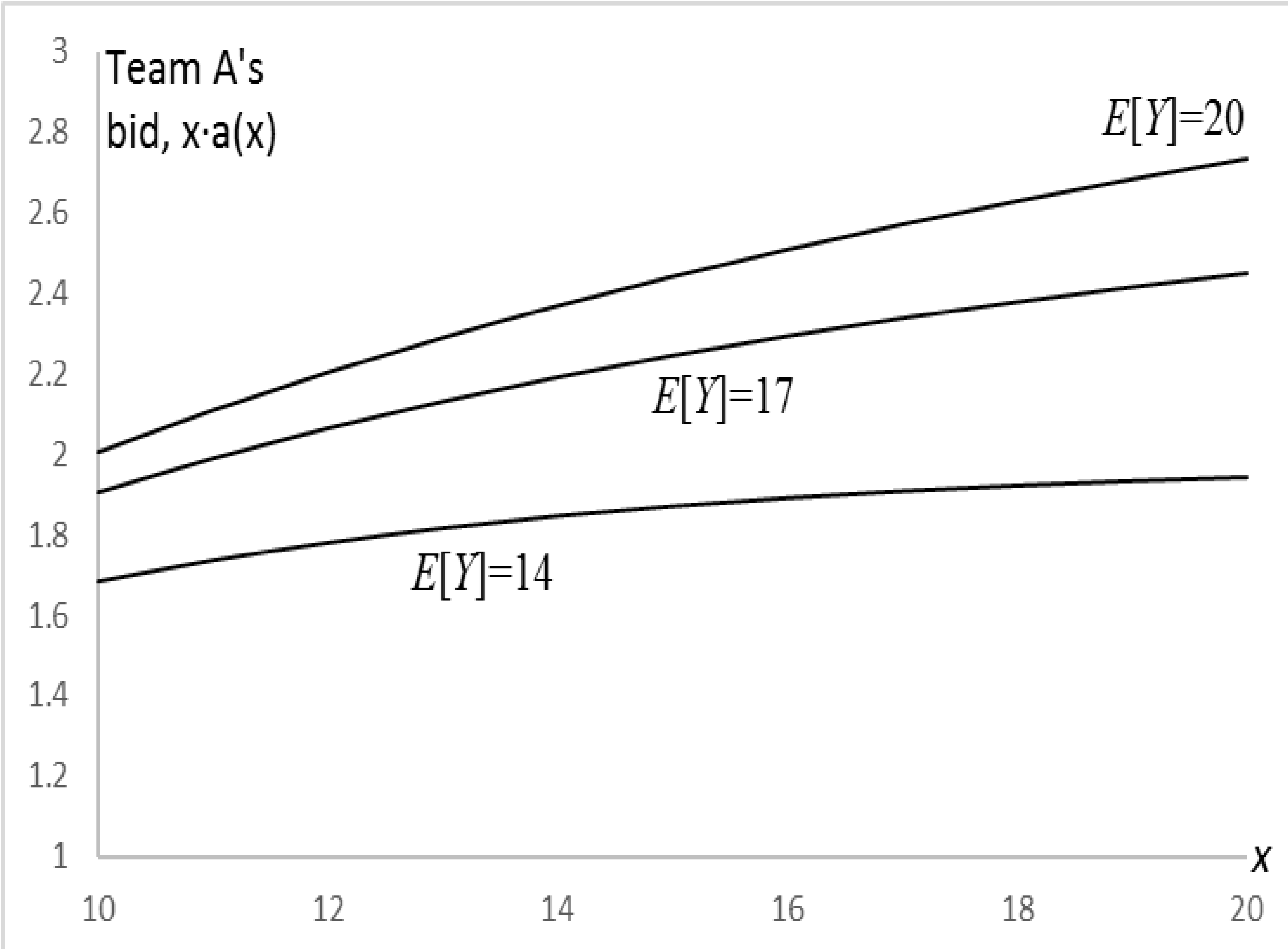
The following table compares the bids corresponding to the two rules.

	$\alpha = 1/2$	$\alpha = 1$
$x = c$	$a_1^* > a_2^*$ if $q(c) + r(c) > 1 + \frac{T+\sqrt{T^2+4TS}}{2S}$, where $S = \left(\frac{\varepsilon}{(1-\varepsilon)(1-q(d))}\right)^2$ and $T = \frac{2p(c)}{p(c)+r(c)}$ $a_1^* < a_2^*$, if $\frac{1}{2}\left(1+\frac{r(c)}{p(c)}\right) < q(c) + r(c) < 1 + \frac{T+\sqrt{T^2+4TS}}{2S}$ $a_1^* = a_2^* = 0$ if $0 < q(c) + r(c) < \frac{1}{2}\left(1+\frac{r(c)}{p(c)}\right)$	$a_1^* > a_2^*$ if $q(c) + r(c) > 1 + \frac{2(1-\varepsilon)(1-q(d))}{\varepsilon}$, $a_1^* < a_2^*$, if $\frac{1}{2}\left(1+\frac{r(c)}{p(c)}\right) < q(c) + r(c) < 1 + \frac{2(1-\varepsilon)(1-q(d))}{\varepsilon}$ $a_1^* = a_2^* = 0$, if $0 < q(c) + r(c) < \frac{1}{2}\left(1+\frac{r(c)}{p(c)}\right)$
$x = d$	$a_1^* > a_2^*$, if $q(d) + r(d) > 1$ $a_1^* < a_2^*$, if $\frac{1}{2}\left(1+\frac{r(d)}{p(d)}\right) < q(d) + r(d) < 1$ $a_1^* = a_2^* = 0$, if $0 < q(d) + r(d) < \frac{1}{2}\left(1+\frac{r(d)}{p(d)}\right)$	$a_1^* > a_2^*$, if $q(d) + r(d) > 1$ $a_1^* < a_2^*$, if $min\left\{1, \frac{1}{2}\left(1+\frac{r(d)}{p(d)}\frac{1+\varepsilon}{1-\varepsilon}\right)\right\} < q(d) + r(d) < 1$ $a_1^* = a_2^* = 0$, otherwise

Exponential distribution of abilities:

$$J = \lambda e^{-a\theta x}\left(\begin{array}{c}(1-e^{-rx})\left(-\frac{e^{-(\lambda+t\theta)x}}{\lambda+t\theta}+\frac{e^{-(\lambda+p+t\theta)x}}{\lambda+p+t\theta}+\frac{e^{-\lambda x}}{\lambda+t(1-\alpha)\theta}\right)\\ +(1-e^{-px})\left(\frac{1-e^{-(\lambda+q+t\theta)x}}{\lambda+q+t\theta}+\frac{e^{t(1-\alpha)\theta x}-e^{-\lambda x}}{\lambda+t(1-\alpha)\theta}-\frac{1-e^{-(\lambda+t\theta)x}}{\lambda+t\theta}\right)\end{array}\right)$$

We maximize J numerically for the parameters used for Rule 1 and plot the results in the following figure.



Team A's bid, x·a(x)
E[Y]=20
E[Y]=17
E[Y]=14
x
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
10
12
14
16
18
20

Presumably, teams who have a player (players) who is good in free kicks from, say, 16-25m will bid higher than ones who have no player with such talent. If Rule 2 is used, that is likely to require the team which lost the bid to kick from a distance from which they are not very good, introducing another positive externality.

In American football, if a game goes into an overtime period ("sudden death", 15 minutes or indefinite), the issue is which team should kick off first and where from (the other team would then start on offence). Bidding for that is a seemingly attractive option (Granot and Gerchak, 2014).