An impossibility result for sport rankings

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Outline

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2017 UEFA European Under-21 Championship I.

One of the last match of the group stage in Group C on 24 June:
Germany vs. Italy

Before	Group	Goals for	Goals against	Points
Germany	C	5	0	6
Italy	C	3	3	3
Slovakia	A	6	3	6

Winners of Groups A, B and C as well as the best runner-up qualifies for the semifinals.
Tie-breaking rules:

- Group: points in head-to-head matches among tied teams
- Best runner-up: points / goal difference / goals scored

2017 UEFA European Under-21 Championship II.

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Tie-breaking rules:

- Group: points in head-to-head matches among tied teams
- Best runner-up: points / goal difference / goals scored

Suppose that **Germany vs. Italy 0-1**

After	Group	Goals for	Goals against	Points
Italy	C	4	3	6
Germany	C	5	1	6
Slovakia	A	6	3	6

But if goal difference would be the group tie-breaking rule, then Slovakia would qualify for the semifinal.

Problem, motivation

Ranking based on paired comparisons

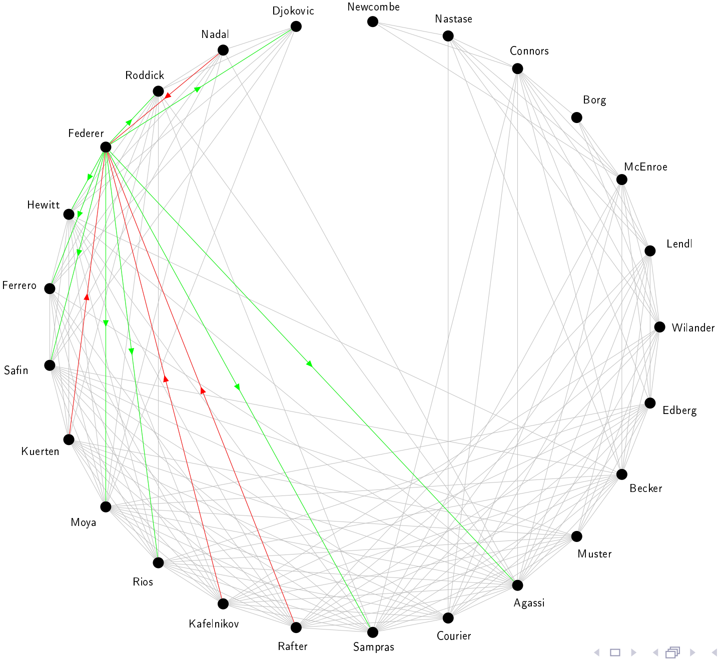
- Sport tournaments
- Psychology
- Journal citations
- Voting on alternatives

Generalized tournament

- Ties and different margins of victory (preference intensities)
- Arbitrary number of matches between players: incomplete (missing) and imbalanced data

Aims

- Better understanding of ranking through an axiomatic approach
- Explore trade-offs among different properties



Federer		
Opponent	Win	Loss
Agassi	8	3
Djokovic	16	15
Ferrero	10	3
Hewitt	18	8
Kafelnikov	2	4
Kuerten	1	2
Moya	7	0
Nadal	10	22
Rafter	0	3
Rios	2	0
Roddick	21	3
Safin	10	2
Sampras	1	0
Sum	106	65

Model setting

General ranking problem $(N, \mathbf{T}) = (N, T^{(1)}, T^{(2)}, \dots, T^{(m)})$

- Set of players: $N = \{X_1, X_2, \dots, X_n\}$
- Tournament matrix of round p: $T^{(p)} \in \mathbb{R}^{n \times n}$ $(p = 1, 2, \dots, m)$
  1. $t_{ij}^{(p)} + t_{ji}^{(p)} = 1$ if players X_i and X_j have played in round p
  2. $t_{ij}^{(p)} = t_{ji}^{(p)} = 0$ otherwise
- Aggregated tournament matrix: $A = \sum_{p=1}^{m} T^{(p)}$

Ranking problem (N, A) or (N, R, M)

- Matches matrix $M = A + A^\top$: symmetric, $m_{ii} = 0$ for all X_i
  $m_{ij} = a_{ij} + a_{ji} = \sum_{p=1}^{m} t_{ij}^{(p)} + t_{ji}^{(p)}$
  $m_{ij} = m_{ji} \in \mathbb{N}$ is the number of matches between X_i and X_j
- Number of matches of player X_i: $d_i = \sum_{X_j \in N} m_{ij}$
- Results matrix $R = T - T^\top$: skew-symmetric, $r_{ii} = 0$ for all X_i
  $r_{ji} = -r_{ij}$ and $r_{ij} \in [-m_{ij}, m_{ij}]$

Rankings and special ranking problems

How to rank the objects?

- $\mathcal{T}^n$ is the set of general ranking problems $(N, \mathbf{T})$ such that $|N| = n$
- General scoring method: $g : \mathcal{T}^n \to \mathbb{R}^n$
- $\mathcal{R}^n$ is the set of ranking problems (N, R, M) such that $|N| = n$
- Scoring method: $f : \mathcal{R}^n \to \mathbb{R}^n$
- Ranking: X_i is ranked weakly above X_j $(X_i \succeq X_j)$ if $f_i(N, R, M) \geq f_j(N, R, M)$

A ranking problem $(N, R, M) \in \mathcal{R}^n$ is called

- *balanced* if $\sum_{X_k \in N} m_{ik} = \sum_{X_k \in N} m_{jk}$ for all $X_i, X_j \in N \Rightarrow \mathcal{R}_B$
- *round-robin* if $m = m_{ij} = m_{k\ell}$ for all $X_i \neq X_j$ and $X_k \neq X_\ell \Rightarrow \mathcal{R}_R$
- *extremal* if $|r_{ij}| \in \{0; m_{ij}\}$ for all $X_i, X_j \in N \Rightarrow \mathcal{R}_E$
  There are only three possibilities: complete win, draw, complete loss

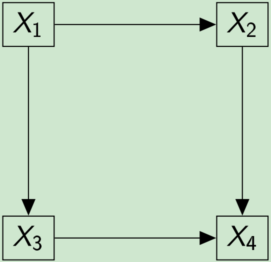
Self-consistency: motivation

Requirements

X_i should be judged better than X_j if one of the following holds:

1. X_i achieves better results against opponents with the same strength
2. X_i achieves the same results against stronger opponents
3. X_i achieves better results against stronger opponents

Example



Implications:

- $X_2 \sim X_3$: they have the same results against the same opponents
- $X_1 \succ X_4$: they have the same opponents, but $r_{12} > r_{42}$ and $r_{13} > r_{43}$
- Finally, only $X_1 \succ (X_2 \sim X_3) \succ X_4$ is allowed

Self-consistency: formal definition

Notations

1. Let $(N, \mathbf{T}) \in \mathcal{T}^n$ be a general ranking problem. The *opponent multiset* of player X_i is O_i, which contains m_{ij} instances of X_j.
2. Let $X_i, X_j \in N$ be two different objects and $h^{(p)} : O_i^{(p)} \leftrightarrow O_j^{(p)}$ be a one-to-one correspondence. Then $\mathfrak{h}^{(p)}$ is given by $X_{\mathfrak{h}^{(p)}(k)} = h^{(p)}(X_k)$.

Self-consistency (SC) [Chebotarev and Shamis, 1997]

Let $(N, \mathbf{T}) \in \mathcal{T}^n$ be a general ranking problem.
Let $X_i, X_j \in N$ be two players and $g : \mathcal{T}^n \to \mathbb{R}^n$ be a general scoring method such that for all $p = 1, 2, \dots, m$ there exists a one-to-one mapping $h^{(p)}$ from $O_i^{(p)}$ onto $O_j^{(p)}$, where $t_{ik}^{(p)} \geq t_{j\mathfrak{h}^{(p)}(k)}^{(p)}$ and $g_k(N, \mathbf{T}) \geq g_{\mathfrak{h}^{(p)}(k)}(N, \mathbf{T})$.
g is called *self-consistent* if $g_i(N, \mathbf{T}) \geq g_j(N, \mathbf{T})$, furthermore, $g_i(N, \mathbf{T}) > g_j(N, \mathbf{T})$ if at least one of the above inequalities is strict.

Additivity of rankings and ranking problems

Order preservation (OP) [González-Díaz et al., 2014]

Let $(N, A), (N, A') \in \mathcal{R}^n$ be two ranking problems where all players have played m matches and $X_i, X_j \in N$ be two different players.
Let $f : \mathcal{R}^n \to \mathbb{R}^n$ be a scoring method such that $f_i(N, A) \geq f_j(N, A)$ and $f_i(N, A') \geq f_j(N, A')$.
f satisfies *order preservation* if $f_i(N, A + A') \geq f_j(N, A + A')$, furthermore, $f_i(N, A + A') > f_j(N, A + A')$ if $f_i(N, A) > f_j(N, A)$ or $f_i(N, A') > f_j(N, A')$.

An example from tennis

Federer $\succeq$ Nadal in 2011 and 2012 $\Rightarrow$ Federer $\succeq$ Nadal in 2011-2012

Remark

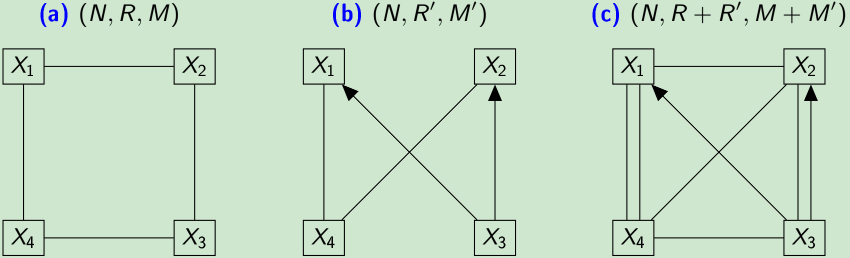
[González-Díaz et al., 2014] show that most scoring methods do not satisfy OP.

Main result: compatibility of *SC* and *OP*

Theorem

No scoring method satisfies order preservation and self-consistency.

Contradiction is proved by an example



SC implies $X_1 \sim X_2 \sim X_3 \sim X_4$ in (N, R, M), $X_1 \sim X_2$ and $X_3 \succ X_4$ in (N, R', M'), so $X_1 \sim X_2$ and $X_3 \succ X_4$ in $(N, R + R', M + M')$ from *OP*

Compare X_1 and X_2: the only difference is that X_1 has a draw against X_4 and X_2 has a draw against X_3, but $X_3 \succ X_4 \Rightarrow X_2 \succ X_1$ according to *SC* ↯

Domain restrictions

Possibility result I.: number of players is at most three

There exists a scoring method satisfying order preservation and self-consistency on the set of ranking problems with at most three objects $\mathcal{R}^n | n \leq 3$.

Remark

Ranking of players is not obvious if $n = 3$.

Another impossibility result: number of players is at least four

No scoring method meets order preservation and self-consistency on the set of balanced and extremal ranking problems with four players $\mathcal{R}_B^4 \cap \mathcal{R}_E^4$.

Possibility result II.: the round-robin case

There exists a scoring method satisfying order preservation and self-consistency on the set of round-robin ranking problems $\mathcal{R}_R$.

Weakening of OP I.

Remark

Order preservation still restricts the matches matrix of the two ranking problems since all players should play m matches in both of them.

Weak order preservation (WOP)

Let $(N, R, M), (N, R', kM) \in \mathcal{R}$ be two ranking problems and $X_i, X_j \in N$ be two different players.
Let $f : \mathcal{R} \to \mathbb{R}^n$ be a scoring method such that $f_i(N, R, M) \geq f_j(N, R, M)$ and $f_i(N, R', kM) \geq f_j(N, R', kM)$.
f satisfies *weak order preservation* if $f_i(N, R + R', M + kM) \geq f_j(N, R + R', M + kM)$, furthermore, $f_i(N, R + R', M + kM) > f_j(N, R + R', M + kM)$ if at least one of the above inequalities is strict.

Corollary

Order preservation implies weak order preservation.

Weakening of *OP* II.

Possibility result III.: weakening of *OP*

There exist scoring methods satisfying weak order preservation and self-consistency.

Remark

Let $(N, R, M), (N, R', M') \in \mathcal{R}_R^n$ be two round-robin ranking problems. Then $M' = kM$.

Corollary

WOP implies OP on the set of round-robin ranking problems $\mathcal{R}_R$.

Observation

Possibility result II. (there exist scoring methods satisfying *OP* and *SC* on the set of round-robin ranking problems $\mathcal{R}_R$) is a consequence of Possibility result III.

Weakening of *SC*

Weak self-consistency (*WSC*)

X_i should be judged better than X_j if one of the following holds:

1. X_i achieves better results against opponents with the same strength
2. X_i achieves the same results against stronger opponents
3. X_i achieves better results against stronger opponents

Corollary

WSC implies SC.

Possibility result IV.

There exist a scoring method satisfying order preservation and and weak self-consistency.

Summary [Csató, 2017b]

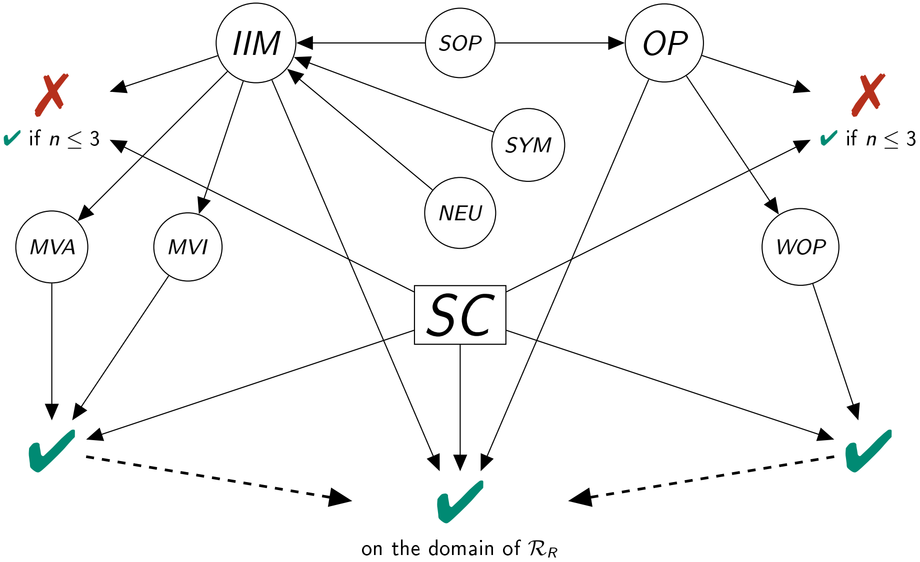
Key points of the research

- Deriving a basic impossibility result on the basis of self-consistency
- Strength of order preservation is not surprising [González-Díaz et al., 2014], but a formal proof was not known
- Effective domain restrictions: $n \leq 3$ or $\mathcal{R}_R$
- Weak order preservation seems to be a good choice

Fields of further investigation

- (Im)possibility theorems with other properties (some new results on this topic can be found in [Csató, 2017a])
- Axiomatic analysis of more (e.g. eigenvector-based) scoring procedu-res
- Characterization: a set of axioms uniquely define a ranking method

An overview of possibility and impossibility results



Thank you for your attention!

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